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Finding the Solution to the Black-Scholes Equation

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by

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Contents

1 Abstract

This paper will explore the solution of the Black-Scholes Equation which is used in mathematical finance. It will derive the solution to the Black-Scholes equation, using the solution of the Heat Equation. This solution can then be used to find the fair price of a European call option. It also includes examples using current stocks.

2 Assumptions

These are assumptions that must be made to use the Black-Scholes equation.

1. The option can only be exercised at the expiration date, as it is a European option.
2. Constant composition returns are normally distributed.
3. Volatility is known and constant.
4. There are efficient markets.
5. Risk-free rate is known and constant.
6. No taxes or transaction costs.
7. There are no dividends during the life of the option.

The European call option is the option to buy the stock at the end of the designated time period set between the buyer and the seller. The call price is a calculated fair price at which to buy the option. This will be further discussed in section 7 of this paper.

3 Variables

This section explains what each variable describes.

- $S=$ Stock price at the beginning of the time period of the option.
- $K=$ Strike price, a price set between the buyer and seller of the option.
- $T-t=$ Expiration date minus start date, the total amount of time until the option is exercised (in years).
- $r=$ Risk-free interest rate.
- $\sigma=$ Volatility of the stock.

4 Explanation of Solution

First, start with the Black-Scholes equation:

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0 \tag{1}$$

Then set $t=T-\frac{\tau}{\frac{1}{2}\sigma^2}$ and solve for τ:

$$\frac{\tau}{\frac{1}{2}\sigma^2}=T-t$$

$$\tau = (T-t)\frac{1}{2}\sigma^2$$

Next set $S = Ke^x$ and solve for x:

$$e^x = \frac{S}{K}$$

$$x = ln\left(\frac{S}{K}\right)$$

With both of these equations, set:

$$V(S,t) = Kv(x,\tau) \tag{2}$$

The next step is to take the first and second derivatives of V with respect to stock price and the first derivative with respect to time:

$$\frac{\partial V}{\partial t} = K\frac{\partial v}{\partial \tau} * \frac{\partial \tau}{\partial t} = K\frac{\partial v}{\partial \tau}\left[(T-t)\frac{1}{2}\sigma^2\frac{\partial}{\partial t}\right] = K\frac{\partial v}{\partial \tau} * \frac{-\sigma^2}{2}$$

$$\frac{\partial V}{\partial S} = K\frac{\partial v}{\partial x} * \frac{\partial x}{\partial S} = K\frac{\partial v}{\partial x}\left[ln\left(\frac{S}{K}\right)\frac{\partial}{\partial S}\right] = K\frac{\partial v}{\partial x} * \frac{1}{S}$$

Using $\frac{\partial x}{\partial S} = \frac{1}{\frac{S}{K}} * \frac{1}{K} = \frac{1}{S}$:

$$\begin{aligned}\frac{\partial^2 V}{\partial S^2} &= \frac{\partial}{\partial S}\left(K\frac{\partial v}{\partial x} * \frac{1}{S}\right)\\ &= K\frac{\partial v}{\partial x} * \frac{-1}{S^2} + K\frac{\partial}{\partial S}\left(\frac{\partial v}{\partial x}\right)\frac{1}{S}\\ &= K\frac{\partial v}{\partial x} * \frac{-1}{S^2} + K\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right)\frac{\partial x}{\partial S} * \frac{1}{S}\\ &= K\frac{\partial v}{\partial x} * \frac{-1}{S^2} + K\frac{\partial^2 v}{\partial x^2} * \frac{1}{S^2}\end{aligned}$$

With these equations, the terminal equation is set to:

$$V(S,T) = max(S-K,0) = max(Ke^x - K, 0)$$

$$V(S,T) = Kv(x,0) \text{ and } v(x,0) = max(e^x - 1, 0)$$

Take the derivatives and plug them back into equation (1):

$$\left(K\frac{\partial v}{\partial \tau} * \frac{-\sigma^2}{2}\right) + \frac{\sigma^2}{2}S^2\left(K\frac{\partial v}{\partial x} * \frac{-1}{S^2} + K\frac{\partial^2 v}{\partial x^2} * \frac{1}{S^2}\right) + rS\left(K\frac{\partial v}{\partial x} * \frac{1}{S}\right) - rKv = 0$$

Simplify the equation by factoring out the K values, canceling out S and S^2:

$$\left(\frac{\partial v}{\partial \tau} * \frac{-\sigma^2}{2}\right) + \frac{\sigma^2}{2}S^2\left(\frac{\partial v}{\partial x} * \frac{-1}{S^2} + \frac{\partial^2 v}{\partial x^2} * \frac{1}{S^2}\right) + rS\left(\frac{\partial v}{\partial x} * \frac{1}{S}\right) - rv = 0$$

$$\left(\frac{\partial v}{\partial \tau} * \frac{-\sigma^2}{2}\right) + \frac{\sigma^2}{2}\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\right) + r\left(\frac{\partial v}{\partial x}\right) - rv = 0$$

Solve for $\frac{\partial v}{\partial \tau}$:

$$\frac{\partial v}{\partial \tau} * \frac{\sigma^2}{2} = \frac{\sigma^2}{2}\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\right) + r\frac{\partial v}{\partial x} - rv$$

Factor out $\frac{\sigma^2}{2}$, let $k = \frac{r}{\frac{\sigma^2}{2}}$ to substitute, and combine like terms:

$$\frac{\partial v}{\partial \tau} = \left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\right) + \frac{r}{\frac{\sigma^2}{2}} * \frac{\partial v}{\partial x} - \frac{r}{\frac{\sigma^2}{2}}v$$

$$\frac{\partial v}{\partial \tau} = \left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\right) + k\frac{\partial v}{\partial x} - kv$$

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv \tag{3}$$

This leaves one parameter, k, that has no dimension. From this, rescale the v equation so that:

$$v = e^{\alpha x + \beta \tau} u(x, \tau) \tag{4}$$

Derive according to x and τ:

$$\frac{\partial v}{\partial \tau} = \beta e^{\alpha x + \beta \tau} u + e^{\alpha x + \beta \tau}\frac{\partial u}{\partial \tau}$$

$$\frac{\partial v}{\partial x} = \alpha e^{\alpha x + \beta \tau} u + e^{\alpha x + \beta \tau}\frac{\partial u}{\partial x}$$

$$\frac{\partial^2 v}{\partial x^2} = \alpha^2 e^{\alpha x + \beta \tau} u + 2\alpha e^{\alpha x + \beta \tau}\frac{\partial u}{\partial x} + e^{\alpha x + \beta \tau}\frac{\partial^2 u}{\partial x^2}$$

Plug these derivatives into equation (3):

$$\beta e^{\alpha x + \beta \tau} u + e^{\alpha x + \beta \tau}\frac{\partial u}{\partial \tau} = \alpha^2 e^{\alpha x + \beta \tau} u + 2\alpha e^{\alpha x + \beta \tau}\frac{\partial u}{\partial x} + e^{\alpha x + \beta \tau}\frac{\partial^2 u}{\partial x^2} + (k-1)\left(\alpha e^{\alpha x + \beta \tau} u + e^{\alpha x + \beta \tau}\frac{\partial u}{\partial x}\right) - k e^{\alpha x + \beta \tau} u$$

Divide by $e^{\alpha x + \beta \tau}$ and combine like terms:

$$\beta u + \frac{\partial u}{\partial \tau} = \alpha^2 u + 2\alpha\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + (k-1)\left(\alpha u + \frac{\partial u}{\partial x}\right) - ku$$

$$\beta u + \frac{\partial u}{\partial \tau} = \alpha^2 u + 2\alpha\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + k\alpha u + k\frac{\partial u}{\partial x} - \alpha u - \frac{\partial u}{\partial x} - ku$$

$$\frac{\partial u}{\partial \tau} = \alpha^2 u + 2\alpha\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + k\alpha u + k\frac{\partial u}{\partial x} - \alpha u - \frac{\partial u}{\partial x} - ku - \beta u$$

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}(k - 1 + 2\alpha) + u(\alpha^2 + k\alpha - \alpha - k - \beta) \tag{5}$$

The coefficients should be equal to zero, meaning that $u = 0$ and $\frac{\partial u}{\partial x} = 0$. Choose $\alpha = \frac{-(k-1)}{2}$ and $\beta = \alpha^2 + (k-1)\alpha - k = \frac{-(k+1)^2}{4}$ then plug into equation (5). This will lead to the basis of the Heat Equation:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\left[k-1+2\left(-\frac{k-1}{2}\right)\right] + u\left[\left(-\frac{k-1}{2}\right)^2 + k\left(-\frac{k-1}{2}\right) - \left(-\frac{k-1}{2}\right) - k - \left(\frac{-(k+1)^2}{4}\right)\right]$$

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}[k-1-(k-1)] + u\left[\left(\frac{k^2-2k+1}{4}\right) - \left(\frac{k^2-k}{2}\right) + \left(\frac{k-1}{2}\right) - k + \left(\frac{k^2+2k+1}{4}\right)\right]$$

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}[0] + u\left[\left(\frac{k^2-2k+1}{4}\right) - \left(\frac{2k^2-2k}{4}\right) + \left(\frac{2k-2}{4}\right) - \frac{4k}{4} + \left(\frac{k^2+2k+1}{4}\right)\right]$$

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + u\left[\frac{k^2-2k+1-2k^2+2k+2k-2-4k+k^2+2k+1}{4}\right]$$

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + u[0]$$

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

$$u_\tau = u_{xx}$$

The initial condition is then transformed into:

$$u(x,0) = max\left(e^{\left(\frac{(k+1)}{2}x\right)} - e^{\left(\frac{(k-1)}{2}x\right)}, 0\right) \quad (6)$$

This leads to the Heat equation solution, which we will transform to use for the Black-Scholes equation:

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} u_o(s) e^{\frac{-(x-s)^2}{4\tau}} ds$$

Make a change of variable so that $s = z\sqrt{2\tau} + x$. The goal is to get the exponent into the form of $\frac{-y^2}{2}$, which is why $z = \frac{x-s}{\sqrt{2\tau}}$, to get the equation of the standard normal deviation. This will then be used later in this derivation to find the final solution. The derivatives of these equations will then be $ds = dx$ and $dx = \sqrt{2\tau}dz$:

$$u(x,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_o(z\sqrt{2\tau} + x) e^{\frac{-z^2}{2}} dz \quad (7)$$

From this transformation, there is a change in equation (6) for the x value:

$$u_o = e^{\frac{k+1}{2}(x+z\sqrt{2\tau})} - e^{\frac{k-1}{2}(x+z\sqrt{2\tau})} \quad (8)$$

It must happen that $u_o > 0$ because the time value cannot be less than 0. So, $x > -\frac{x}{\sqrt{2\tau}}$ which transforms the base of the domain of equation (7):

$$u(x,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\frac{k+1}{2}(x+z\sqrt{2\tau})} e^{-\frac{z^2}{2}} - e^{\frac{k-1}{2}(x+z\sqrt{2\tau})} e^{-\frac{z^2}{2}} dz$$

$$u(x,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\frac{k+1}{2}(x+z\sqrt{2\tau})} e^{-\frac{z^2}{2}} dz - \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\frac{k-1}{2}(x+z\sqrt{2\tau})} e^{-\frac{z^2}{2}} dz$$

$$u(x,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\frac{k+1}{2}(x+z\sqrt{2\tau})-\frac{z^2}{2}} dz - \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\frac{k-1}{2}(x+z\sqrt{2\tau})-\frac{z^2}{2}} dz \qquad (9)$$

After the split of the integral, take the first integral and complete the square of the exponent:

$$\begin{aligned} \frac{k+1}{2}(x+z\sqrt{2\tau}) - \frac{z^2}{2} &= -\frac{1}{2}\left[z^2 - z\sqrt{2\tau}(k+1)\right] + \frac{x(k+1)}{2} \\ &= -\frac{1}{2}\left[z^2 - z\sqrt{2\tau}(k+1) + \frac{\tau}{2}(k+1)^2\right] + \frac{x(k+1)}{2} - \left[-\frac{\tau(k+1)^2}{4}\right] \\ &= -\frac{1}{2}\left[z - \sqrt{\frac{\tau}{2}}(k+1)\right]^2 + \frac{x(k+1)}{2} + \frac{\tau(k+1)^2}{4} \end{aligned}$$

Plug this value back into the first integral of equation (9). This value is the exponent of e. The last two parts of the exponent do not have z values, so they go in front of the integral:

$$\frac{e^{\frac{x(k+1)}{2}+\frac{\tau(k+1)^2}{4}}}{\sqrt{2\pi}} \int_{\frac{-x}{\sqrt{2\tau}}}^{\infty} e^{-\frac{1}{2}\left[z-\sqrt{\frac{\tau}{2}}(k+1)\right]^2} dz$$

Set $y = z - \sqrt{\frac{\tau}{2}}(k+1)$, $dy = dz$, and $z = \frac{-x}{\sqrt{2\tau}}$ which in turn changes the domain once again:

$$\frac{e^{\frac{x(k+1)}{2}+\frac{\tau(k+1)^2}{4}}}{\sqrt{2\pi}} \int_{\frac{-x}{\sqrt{2\tau}}-\sqrt{\frac{\tau}{2}}(k+1)}^{\infty} e^{-\frac{y^2}{2}} dy$$

Equation (9) then becomes:

$$u(x,\tau) = \frac{e^{\frac{x(k+1)}{2}+\frac{\tau(k+1)^2}{4}}}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x}{\sqrt{2\tau}}+\sqrt{\frac{\tau}{2}}(k+1)} e^{-\frac{y^2}{2}} dy - \frac{e^{\frac{x(k-1)}{2}+\frac{\tau(k-1)^2}{4}}}{\sqrt{2\pi}} \int_{\frac{-x}{\sqrt{2\tau}}-\sqrt{\frac{\tau}{2}}(k+1)}^{\infty} e^{-\frac{y^2}{2}} dy \qquad (10)$$

The area under normal curve formula from $-\infty \to d$ is:

$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{y^2}{2}} dy$$

where,

$$d = \frac{x}{\sqrt{2\tau}} + \sqrt{\frac{\tau}{2}}(k+1)$$

$-\infty \to d$ is the same as $-d \to \infty$. The value of d_2 is the same as d_1 with the exception that $(k+1)$ is $(k-1)$, so:

$$d_1 = \frac{x}{\sqrt{2\tau}} + \sqrt{\frac{\tau}{2}}(k+1)$$

$$d_2 = \frac{x}{\sqrt{2\tau}} + \sqrt{\frac{\tau}{2}}(k-1)$$

Plug N into the equation (10):

$$u(x,\tau) = e^{\frac{x(k+1)}{2}+\frac{\tau(k+1)^2}{4}}N(d_1) - e^{\frac{x(k-1)}{2}+\frac{\tau(k-1)^2}{4}}N(d_2) \tag{11}$$

Now plug α, β, and equation (11) into equation (4):

$$v(x,\tau) = e^{\frac{-x(k-1)}{2}-\frac{\tau(k+1)^2}{4}}u(x,\tau)$$

$$v(x,\tau) = e^{\frac{-x(k-1)}{2}-\frac{\tau(k+1)^2}{4}} * \left[e^{\frac{(k+1)x}{2}+\frac{\tau(k+1)^2}{4}}N(d_1) - e^{\frac{x(k-1)}{2}+\frac{\tau(k-1)^2}{4}}N(d_2)\right]$$

The exponents will then cancel out to get a simple equation of:

$$v(x,\tau) = e^{x}N(d_1) - e^{-k\tau}N(d_2)$$

There are two values from earlier that need to be plugged back in: $x = ln(S/K)$ and $\tau = \frac{1}{2}\sigma^2(T-t)$ to get:

$$v(x,\tau) = e^{ln(S/K)}N(d_1) - e^{-\frac{k}{2}\sigma^2(T-t)}N(d_2)$$

$$v(x,\tau) = \frac{S}{K}N(d_1) - e^{-\frac{k}{2}\sigma^2(T-t)}N(d_2) \tag{12}$$

Plug these values into the d-values as well:

$$\begin{aligned} d_1 &= \frac{ln\left(\frac{S}{K}\right)}{\sqrt{2\left(\frac{1}{2}\sigma^2(T-t)\right)}} + \sqrt{\frac{\frac{1}{2}\sigma^2(T-t)}{2}}(k+1) \\ &= \frac{ln\left(\frac{S}{K}\right)}{\sigma\sqrt{T-t}} + \frac{\sigma}{2}\sqrt{T-t}(k+1) \\ &= \frac{ln\left(\frac{S}{K}\right)}{\sigma\sqrt{T-t}} + \frac{\frac{\sigma^2}{2}(T-t)(k+1)}{\sigma\sqrt{T-t}} \\ &= \frac{ln\left(\frac{S}{K}\right)\left(\frac{\sigma^2}{2}k + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \end{aligned}$$

The risk-free interest rate is equal to $r = \frac{k}{2}\sigma^2$. So then equation (12) and the d-value become:

$$v(x,\tau) = \frac{S}{K}N(d_1) - e^{-r(T-t)}N(d_2) \tag{13}$$

$$d_1 = \frac{ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

Equation (13) is then plugged back into equation (2):

$$\begin{aligned} V(S,t) &= K\frac{S}{K}N(d_1) - Ke^{-r(T-t)}N(d_2) \\ &= SN(d_1) - Ke^{-r(T-t)}N(d_2) \end{aligned}$$

We then have the solution to the Black-Scholes Equation:

$$V(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2) \tag{14}$$

where,

$$d_1 = \frac{ln\big(\frac{S}{K}\big) + \big(r + \frac{\sigma^2}{2}\big)(T-t)}{\sigma\sqrt{T-t}} \tag{15}$$

$$d_2 = \frac{ln\big(\frac{S}{K}\big) + \big(r - \frac{\sigma^2}{2}\big)(T-t)}{\sigma\sqrt{T-t}} \tag{16}$$

5 Standard Normal Distribution

The derivation of the Black-Scholes equation uses lognormal probabilities to calculate the solution equation. This uses the standard normal distribution of the values, given by the equation:

$$d = \frac{ln\big(\frac{S}{K}\big) + \big(r + \frac{\sigma^2}{2}\big)(T-t)}{\sigma\sqrt{T-t}}$$

This equation is then used within $N(d)$. The standard normal distribution is a probability bell-curve. The equation of the curve is:

$$y = \frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

Where μ is the mean and σ is the standard deviation. To find the value of the lognormal probability, take the d value and find the value for $N(d)$ within a cumulative area under the Standard Normal Distribution table.

6 Examples

6.1 Example from Jørgen Veisdal

The first example found was by Jørgen Veisdal through the website, *Medium*. He looked at the Tesla stock (TSLA). When he looked at it, the stock price was at \$245. He then set the strike price to \$294 to be taken after 101 days. The volatility of the stock was 38% and the interest rate was 2.12%. Then I take these values and plug them into the Black-Scholes Equation solution:

$$d_1 = \frac{ln\big(\frac{S}{K}\big) + \big(r + \frac{\sigma^2}{2}\big)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_1 = \frac{ln\big(\frac{245}{294}\big) + \big(0.0212 + \frac{0.38^2}{2}\big)\big(\frac{101}{365}\big)}{0.38\sqrt{\frac{101}{365}}}$$

$$d_1 = \frac{-0.18232 + 0.02584}{0.19989}$$

$$d_1 = -0.78280$$

$$N(d_1) = N(-0.78) = 0.2177$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

$$d_2 = -0.78280 - 0.38\sqrt{\frac{101}{365}}$$

$$d_2 = -0.98269$$

$$N(d_2) = N(-0.98) = 0.1635$$

$$C(S,t) = S * N(d_1) - Ke^{-r(T-t)}N(d_2)$$

$$C(S,t) = 245 * (0.2177) - 294 * e^{-0.0212(\frac{101}{365})} * (0.1635)$$

$$C(S,t) = 53.34 - (294)(0.99415)(0.1635)$$

$$C(S,t) = 53.34 - 47.79$$

$$C(S,t) = \$5.55$$

6.2 Example from Kevin Bracker

The next example is taken from a video on *YouTube*, created by Kevin Bracker. The stock price was at \$62 with a strike price of \$60. The option would then be taken over 40 days with a volatility of 32% and an interest rate of 4%:

$$d_1 = \frac{ln\big(\frac{S}{K}\big) + \big(r + \frac{\sigma^2}{2}\big)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_1 = \frac{ln\big(\frac{62}{60}\big) + \big(0.04 + \frac{0.32^2}{2}\big)\big(\frac{40}{365}\big)}{0.32\sqrt{\frac{40}{365}}}$$

$$d_1 = 0.40388$$

$$N(d_1) = N(0.40) = 0.6554$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

$$d_2 = 0.40388 - 0.32\sqrt{\frac{40}{365}}$$
$$d_2 = 0.29795$$

$$N(d_2) = N(0.30) = 0.6179$$

$$C(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$
$$C(S,t) = 62 * (0.6554) - 60 * e^{-0.04(\frac{40}{365})} * (0.6179)$$
$$C(S,t) = \$3.72$$

6.3 Pfizer Example

For this example, I tracked the stock of Pfizer Inc. (PFE) and used the Black-Scholes solution to calculate the call price. The stock price, on November 6, 2019, was \$37.25, I then set the strike price to \$39 to be taken over 30 days. The volatility being 22% and an interest rate of 4.2%:

$$d_1 = \frac{ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$
$$d_1 = \frac{ln\left(\frac{37.25}{39}\right) + \left(0.042 + \frac{0.22^2}{2}\right)\left(\frac{30}{365}\right)}{0.22\sqrt{\frac{30}{365}}}$$
$$d_1 = -0.6416$$

$$N(d_1) = N(-0.64) = 0.2611$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$
$$d_2 = -0.6416 - 0.22\sqrt{\frac{30}{365}}$$
$$d_2 = -0.7047$$

$$N(d_2) = N(-0.70) = 0.2420$$

$$C(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$
$$C(S,t) = 37.25 * (0.2611) - 39 * e^{-0.042(\frac{30}{365})} * (0.2420)$$
$$C(S,t) = \$0.32$$

Pfizer is not a volatile stock, thus resulting in a low call price for this option.

6.4 Bitcoin Example

The second stock I tracked was Bitcoin USD (BTC-USD). The starting stock price was \$9,355.025 and I set the strike price to \$10,000. Over the time period of 180 days, the volatility was 36.87% and an interest rate of 2.25%:

$$d_1 = \frac{ln\big(\frac{S}{K}\big) + \big(r + \frac{\sigma^2}{2}\big)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_1 = \frac{ln\big(\frac{9355.025}{10000}\big) + \big(0.0225 + \frac{0.3687^2}{2}\big)\big(\frac{180}{365}\big)}{0.3687\sqrt{\frac{180}{365}}}$$

$$d_1 = -0.0852$$

$$N(d_1) = N(-0.08) = 0.4681$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

$$d_2 = -0.0852 - 0.3687\sqrt{\frac{180}{365}}$$

$$d_2 = -0.3441$$

$$N(d_2) = N(-0.34) = 0.3669$$

$$C(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$

$$C(S,t) = 9255.025 * (0.4681) - 10000 * e^{-0.0225(\frac{180}{365})} * (0.3669)$$

$$C(S,t) = \$750.58$$

Since Bitcoin is a very volatile stock, the call price would be very large which connects to this example.

6.5 Apple Example

The last stock I tracked was Apple INC (APPL). With a stock price of \$256.47, I set the strike price to \$270 to be exercised after 90 days. The volatility was 27.5% and an interest rate of 8.07%:

$$d_1 = \frac{ln\big(\frac{S}{K}\big) + \big(r + \frac{\sigma^2}{2}\big)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_1 = \frac{ln\big(\frac{256.47}{270}\big) + \big(0.0807 + \frac{0.275^2}{2}\big)\big(\frac{90}{365}\big)}{0.275\sqrt{\frac{90}{365}}}$$

$$d_1 = -0.1625$$

$$N(d_1) = N(-0.16) = 0.4364$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$
$$d_2 = -0.1625 - 0.275\sqrt{\frac{90}{365}}$$
$$d_2 = -0.2991$$

$$N(d_2) = N(-0.30) = 0.3821$$

$$C(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$
$$C(S,t) = 256.47 * (0.4364) - 270 * e^{-0.0807(\frac{90}{365})} * (0.3821)$$
$$C(S,t) = \$10.79$$

Apple is a decently volatile stock thus resulting in the $10 call price for the option presented.

7 What It All Means

What is the point of the Black-Scholes equation and solution? While doing a European option, the idea of Black-Scholes is to find the fairest price for one share of a stock. Since it is a European option, this means that the option can only be exercised at the end of the time period set by the buyer and seller. While this paper focuses on the call option, there are few differences for a put option. The put option is the option to sell the stock after a set period of time. The put option formula is $P(S,t) = Ke^{-r(T-t)}N(-d_2) - SN(-d_1)$.

The call option is the option to buy the stock after a set period of time. The call option formula is $V(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2)$. This translates to the stock price multiplied by the probability of normal distribution for the d_1 value subtracted by the strike price multiplied by e, which is discounting the strike price back to the initial day, and multiplied by the normal distribution for the d_1 value. The normal distribution is explained in section 5 of this paper. From the assumptions stated in section 2, the volatility and risk-free rate are known and constant. The stock price is the value of the share at the start of the option. The values left within the equation are the time and strike price, these are agreed upon by the buyer and seller of the share of stocks.

The call price calculated from the solution to the Black-Scholes equation is the fair price for one share of a stock. When paying the seller this amount, the buyer is purchasing the right to buy the stock at the end of the specified time period at the strike price. So, if the stock price exceeds the strike price plus the call price, the buyer will make a profit from the option.

This can be better understood by using the example from subsection 6.1. The buyer has the right to buy the stock in 101 days at $294 at a cost of $5.55 per share. For 100 shares of the stock of Tesla, this means that the buyer pays the seller $555 for the call option at the start date. At the end date, if the stock, originating at $245, is below $299.55 (294+5.55), then the buyer has the right to not buy the stock. If they do not, then the seller keeps their shares of the Tesla stock and the buyer loses the $555 paid to the seller at the beginning of the option. If the stock only reaches $299.55, the buyer and seller will break even. If the stock exceeds $299.55 the buyer will profit from the option. For instance, the stock reaches $400 at the end of the time period, then the seller keeps the $555 given to them by the buyer at the beginning of the time period, but has to sell the stock at $294 per stock, even though it is valued at $400 per stock.

The example explained in the last paragraph shows that using the solution to the Black-Scholes equation can be extremely risky. As European options can only be exercised at the end of the time period, one will never know where the stock price will be at in the end. Everything is dependent on the volatility of the stock. The more volatile, the higher the probability the stock price will change a great deal. That is why when the call option is calculated, from the Black-Scholes solution, it is setting a fair price to buy the option of the stock based on many factors, including the volatility.

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